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On Normal Pointsystems of Hermite–Fejér Interpolation of Arbitrary Order[1]

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Necessary conditions of normal pointsystems for Hermite–Fejér interpolation of arbitrary (even) order are given. In particular, one of the main results in this paper is: If a pointsystem consists of the zeros of orthogonal polynomials with respect to a weight w on $[-1, 1]$ and is always normal for Hermite–Fejér interpolation of arbitrary (even) order, then $w(x) \sim (1-x^2)^{-1/2}$.

1. INTRODUCTION AND MAIN RESULTS

Let $m, n \in \mathbb{N}$ and

$$X: 1 = x_{0n} \geqslant x_{1n} > x_{2n} > \cdots > x_{nn} \geqslant x_{n+1,n} = -1. \tag{1.1}$$

In what follows we shall often omit the superfluous notations. The symbols $c, c_1, \ldots$ will stand for positive constants, being independent of variables and indices, unless otherwise indicated; their value may be different at different occurrences, even in subsequent formulas. $\mathbb{N}_2$ stands for the set of even integers of $\mathbb{N}$. Let w be a weight on $[-1, 1]$ and $P_n(x) = P_n(w, x) = \gamma_n(w)\, x^n + \cdots (\gamma_n(w) > 0)$ the nth orthonormal polynomial with respect to w on $[-1, 1]$ ($P_0 = 1$). $\lambda_n(x) = \lambda_n(w, x) = [\sum_{k=0}^{n-1} P_k(w, x)^2]^{-1}$ is the Christoffel function. $|I|$ means the Lebesgue measure of the set I. $\|\cdot\|$ denotes the uniform norm on $[-1, 1]$. Denote by $\mathbf{P}_N$ the set of polynomials of degree at most N and by $A_{jk} = A_{jknm}(X)$ the fundamental polynomials for Hermite interpolation, i.e., $A_{jk} \in \mathbf{P}_{mn-1}$ satisfy

$$A_{jk}^{(p)}(x_q) = \delta_{jp}\delta_{kq}, \qquad p, j = 0, 1, \ldots, m-1, \quad q, k = 1, 2, \ldots, n. \tag{1.2}$$

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The Hermite–Fejér interpolation for $f \in C[-1, 1]$ is given by

$$H_{nm}(f, x) = \sum_{k=1}^{n} f(x_k)\, A_{0k}(x).$$

To give an explicit formula for A_{jk} set

$$\ell_k(x) = \frac{\omega_n(x)}{\omega_n'(x_k)(x-x_k)},$$

$$\omega_n(x) = (x-x_1)(x-x_2)\cdots(x-x_n),$$

$$b_{ik} = \frac{1}{i!}\,[\ell_k(x)^{-m}]^{(i)}_{x=x_k},$$

$$i = 0, 1, ..., m-1, \quad k = 1, 2, ..., n, \tag{1.3}$$

$$B_{jk} = \sum_{i=0}^{m-j-1} b_{ik}(x-x_k)^i,$$

$$j = 0, 1, ..., m-1, \quad k = 1, 2, ..., n. \tag{1.4}$$

Then we have [7]

$$A_{jk}(x) = \frac{1}{j!}\,(x-x_k)^j\, B_{jk}(x)\, \ell_k(x)^m,$$

$$j = 0, 1, ..., m-1, \quad k = 1, 2, ..., n.$$

We also introduce the notations

$$\Omega_n(x) = 2^n \omega_n(x), \qquad \delta_n(x) = \frac{(1-x^2)^{1/2}}{n},$$

$$\Delta_n(x) = \delta_n(x) + \frac{1}{n^2}, \qquad v(x) = (1-x^2)^{-1/2}.$$

A pointsystem X is said to be normal for Hermite–Fejér interpolation $H_{nm}(X)$, $m \in \mathbb{N}_2$, if

$$A_{0knm}(x) \geqslant 0, \quad x \in [-1, 1], \quad x \in [-1, 1], \quad k = 1, 2, ..., n, \quad n = 1, 2, ..., \tag{1.5}$$

This paper will give necessary conditions for a pointsystem to be normal for $H_{nm}(X)$ for every $m \in \mathbb{N}_2$. The first main result is

THEOREM 1.1. *If a pointsystem X satisfies*

$$\max_{1\leqslant k\leqslant n} \|A_{0knm}(x)\| \leqslant c_m, \qquad \forall m \in \mathbb{N}_2, \tag{1.6}$$

with

$$\limsup_{m\to\infty} (c_m)^{1/m} < +\infty, \tag{1.7}$$

then

(a) *we have*

$$\|(x-x_k)\, \ell_k(x)\| \leqslant c\Delta_n(x_k), \qquad k=1, 2, \ldots, n; \tag{1.8}$$

(b) *we have*

$$\|(x-x_k)\, \ell_k(x)\| \sim \Delta_n(x_k), \qquad k=1, 2, \ldots, n; \tag{1.9}$$

(c) *we have*

$$\sup_n \|\Omega_n\| < +\infty \tag{1.10}$$

and

$$|\Omega_n'(x_k)| \sim \Delta_n(x_k)^{-1}, \qquad k=1, 2, \ldots, n; \tag{1.11}$$

(d) *the relation* (1.10) *is true and*

$$\max_{x_{k+1}\leqslant x\leqslant x_k} |\Omega_n(x)| \geqslant c, \qquad k=1, 2, \ldots, n-1; \tag{1.12}$$

(e) *the relation* (1.10) *is true and the inequality*

$$\limsup_{k\to\infty} |\Omega_{n_k}(x)| \geqslant c, \qquad a.e. \quad x\in[-1, 1], \tag{1.13}$$

holds for every subsequence $\mathbb{N}^* = \{n_k\}_{k=1}^{\infty}$.

The second main result is

THEOREM 1.2. *Let w be a weight on* $[-1, 1]$ *and X the zeros of the orthonormal polynomial* $P_n(w, x)$. *If* (1.6) *and* (1.7) *are true then*

(A) *we have*

$$\|(x-x_k)\, \ell_k(x)\| \leqslant c\delta_n(x_k), \qquad k=1, 2, \ldots, n; \tag{1.14}$$

(B) *we have*

$$\|(x-x_k)\,\ell_k(x)\| \sim \delta_n(x_k), \qquad k=1,2,\ldots,n; \tag{1.15}$$

(C) *we have*

$$\sup_n \|P_n(w)\| < +\infty \tag{1.16}$$

and

$$|P'_n(w, x_k)| \sim \delta_n(x_k)^{-1}, \qquad k=1,2,\ldots,n; \tag{1.17}$$

(D) *the relation* (1.16) *is true and*

$$\max_{x_{k+1} \leqslant x \leqslant x_k} |P_n(w, x)| \geqslant c, \qquad k=0,1,\ldots,n; \tag{1.18}$$

(E) *the relation* (1.16) *is true and the inequality*

$$\limsup_{k\to\infty} |P_{n_k}(w, x)| \geqslant c, \qquad a.e. \quad x \in [-1, 1], \tag{1.19}$$

holds, for every subsequence $\mathbb{N}^*$*;*

(F) *we have*

$$w \sim v, \qquad a.e.; \tag{1.20}$$

(G) *we have*

$$\lambda_n(w, x) \sim \frac{1}{n}, \qquad x \in [-1, 1].$$

The last main result is

THEOREM 1.3. *If a pointsystem X is normal for* $H_{nm}(X)$ *for every* $m \in \mathbb{N}_2$ *then Statements* (a)–(e) *hold and the relation*

$$\lim_{n\to\infty} \|H_{nm}(X, f) - f\| = 0, \qquad \forall \in C[-1, 1] \tag{1.21}$$

holds for every $m \in \mathbb{N}_2$. *Furthermore, if X consists of the zeros of the orthonormal polynomial* $P_n(w, x)$ *and is normal, then Statements* (A)–(G) *hold.*

2. AUXILIARY LEMMAS

First we state some known results needed below.

LEMMA A [8, Lemma 3; 7, Theorem 2.1]. *If $m-j$ is odd and $j<i<m$ then*

$$|A_{jk}(x)| \geqslant \frac{i!}{j!} d_k^{j-i} |A_{ik}(x)|, \qquad x \in [-1, 1], \quad k=1, 2, ..., n, \tag{2.1}$$

where $d_k = \max\{|x_k - x_{k-1}|, |x_k - x_{k+1}|\}$.

LEMMA B [7, Lemma 5.1]. *Let* $x_{kn} = \cos\theta_{kn}$. *If*

$$\theta_{k+1,n} - \theta_{kn} \geqslant \frac{c_1}{n}, \qquad k=1, 2, ..., n-1, \tag{2.2}$$

and

$$\theta_{k+1,n} - \theta_{kn} \leqslant \frac{c_2}{n}, \qquad k=0, 1, ..., n, \tag{2.3}$$

then

$$|x_j - x_k| \sim \frac{|j-k| \min\{j+k, 2n+2-j-k\}}{n^2},$$

$$j \neq k, \quad 1 \leqslant j, k \leqslant n, \tag{2.4}$$

and

$$d_k \sim \Delta_n(x_k) \sim \frac{\min\{k, n+1-k\}}{n^2}, \qquad k=1, 2, ..., n. \tag{2.5}$$

LEMMA C [7, Lemma 5.2]. *If* (2.2) *and* (2.3) *are true and*

$$|\ell_k'(x_k)| \leqslant c\Delta_n(x_k)^{-1}, \qquad k=1, 2, ..., n, \tag{2.6}$$

then

$$|b_{ik}| \leqslant c\Delta_n(x_k)^{-i}, \qquad k=1, 2, ..., n, \quad i=0, 1, ... \tag{2.7}$$

Next, we give some auxiliary lemmas, which are of independent interest.

LEMMA 2.1. *If* (1.8) *is true, we have* (2.2), (2.3),

$$\|\ell_k\| \leqslant c, \qquad k = 1, 2, \ldots, n, \tag{2.8}$$

and

$$\left\| \sum_{k=1}^{n} |\ell_k|^p \right\| \leqslant \begin{cases} c \ln n, & p = 1, \\ c, & p > 1. \end{cases} \tag{2.9}$$

Proof. We denote by c_0 the constant c in (1.8) and may assume $c_0 \geqslant 1$. Let k, $1 \leqslant k \leqslant n$, be fixed and let $|\ell_k(\xi)| = \|\ell_k\|$, $\xi \in [-1, 1]$. If $|x_k - \xi| \geqslant c_0 \Delta_n(x_k)$ then by (1.8) $\|\ell_k\| = |\ell_k(\xi)| \leqslant 1$. Let us consider the case when $|x_k - \xi| < c_0 \Delta_n(x_k)$. By Bernstein Inequality $|\ell_k'(\xi)| \leqslant c\Delta_n(\xi)^{-1} |\ell_k(\xi)|$, whence according to (1.8) we have

$$|\ell_k(\xi) + (\xi - x_k)\, \ell_k'(\xi)| \leqslant c_3 \Delta_n(\xi)^{-1}\, c_0 \Delta_n(x_k) = \frac{c_0 c_3 \Delta_n(x_k)}{\Delta_n(\xi)}. \tag{2.10}$$

We claim that

$$d := \frac{\Delta_n(x_k)}{\Delta_n(\xi)} \leqslant c_4 = 5c_0. \tag{2.11}$$

In fact, if $|\xi| \leqslant |x_k|$ then $\Delta_n(x_k) \leqslant \Delta_n(\xi)$ and hence $d \leqslant 1$. Now let us prove (2.11) for $|\xi| > |x_k|$. By the mean value theorem for the derivative

$$\begin{aligned} \left| \frac{\Delta_n(x_k) - \Delta_n(\xi)}{\Delta_n(\xi)} \right| &= \frac{1}{n\Delta_n(\xi)} |(1 - x_k^2)^{1/2} - (1 - \xi^2)^{1/2}| \\ &= \frac{|\eta|\, |x_k - \xi|}{n(1 - \eta^2)^{1/2}\, \Delta_n(\xi)} \leqslant \frac{c_0}{n(1 - \eta^2)^{1/2}} \cdot \frac{\Delta_n(x_k)}{\Delta_n(\xi)}, \end{aligned}$$

where $|\xi| > |\eta| > |x_k|$. Using the symbol d in (2.11) the above inequality becomes

$$d - 1 \leqslant \frac{c_0}{n(1 - \eta^2)^{1/2}}\, d. \tag{2.12}$$

We separate two cases.

Case 1. $c_0/n(1 - \eta^2)^{1/2} \leqslant 1/2$. In this case by (2.12) we have $d - 1 \leqslant d/2$ and hence $d \leqslant 2$.

Case 2. $c_0/n(1-\eta^2)^{1/2}>1/2$. In this case

$$(1-\xi^2)^{1/2}<(1-\eta^2)^{1/2}<\frac{2c_0}{n}$$

and hence

$$\begin{aligned}1-x_k^2&=1-\xi^2+\xi^2-x_k^2\leqslant 1-\xi^2+2\,|\xi-x_k|\\&\leqslant\frac{4c_0^2}{n^2}+2c_0\Delta_n(x_k)\leqslant\frac{2c_0(1-x_k^2)^{1/2}}{n}+\frac{6c_0^2}{n^2}.\end{aligned}$$

We rewrite the above inequality as

$$[(1-x_k^2)^{1/2}]^2-\frac{2c_0}{n}[(1-x_k^2)^{1/2}]-\frac{6c_0^2}{n^2}\leqslant 0$$

and solve it to get $(1-x_k^2)^{1/2}\leqslant 4c_0/n$. Thus

$$\Delta_n(x_k)\leqslant\frac{4c_0}{n^2}+\frac{1}{n^2}\leqslant\frac{5c_0}{n^2}.$$

Since $\Delta_n(\xi)\geqslant 1/n^2$, by (2.11) we conclude $d\leqslant 5c_0$. This proves (2.11) and hence (2.10) becomes

$$|\ell_k(\xi)+(\xi-x_k)\,\ell_k'(\xi)|\leqslant c_5=5c_0^2c_3. \tag{2.13}$$

If $|\xi|<1$ then $\ell_k'(\xi)=0$ and (2.13) yields $\|\ell_k\|=|\ell_k(\xi)|\leqslant c_5$. If $|\xi|=1$ then, noticing that ℓ_k has only real zeros which are all in $[-1,1]$, $\ell_k(\xi)[(\xi-x_k)\,\ell_k'(\xi)]\geqslant 0$. The above inequality together with (2.13) also gives $\|\ell_k\|=|\ell_k(\xi)|\leqslant c_5$. This, proves (2.8).

Inequality (2.2) and (2.3) directly follow from (2.8) by [1, Theorem 1].

To prove (2.9) let $x\in[-1,1]$ be fixed and $|x-x_j|=\min_{1\leqslant k\leqslant n}|x-x_k|$. Suppose without loss of generality that $j\leqslant n/2$. Then

$$\min\{k+j,\,2n+2-k-j\}\geqslant(k+j)/3. \tag{2.14}$$

Clearly $\|\ell_j\|\leqslant c$ by (2.8). For $k\neq j$ by (1.8), (2.4), (2.5), and (2.14)

$$|\ell_k(x)|\leqslant\frac{c_0\Delta_n(x_k)}{|x-x_k|}\leqslant\frac{c\Delta_n(x_k)}{|x_j-x_k|}\leqslant\frac{c\min\{k,\,n+1-k\}}{|k^2-j^2|}.$$

Then

$$\sum_{k\neq j}|\ell_k(x)|^p\leqslant c^p\sum_{k\neq j}\left[\frac{\min\{k,\,n+1-k\}}{|k^2-j^2|}\right]^p:=c^p[S_1+S_2+S_3+S_4],$$

where

$$S_i = \sum_{k \in K_i} \left[\frac{\min\{k, n+1-k\}}{|k^2 - j^2|} \right]^p, \qquad i = 1, 2, 3, 4,$$

and

$$K_1 := \{k : k \leqslant \tfrac{1}{2} j\}, \qquad K_2 := \{k : \tfrac{1}{2} j < k \leqslant \tfrac{3}{2} j, k \neq j\},$$
$$K_3 := \{k : \tfrac{3}{2} j < k \leqslant \tfrac{3}{4} n\}, \qquad K_4 := \{k : \tfrac{3}{4} n < k \leqslant n\}.$$

It is easy to see that

$$S_1 \leqslant c \sum_{k \in K_1} \left[\frac{j}{j^2} \right]^p = c \sum_{k \in K_1} j^{-p} \leqslant c j^{1-p} \leqslant c;$$

$$S_2 \leqslant c \sum_{k \in K_2} \left[\frac{j}{j\,|k-j|} \right]^p = c \sum_{k \in K_2} |k-j|^{-p} \leqslant \begin{cases} c \ln j, & p = 1, \\ c, & p > 1. \end{cases}$$

For $k \in K_3$ we have $k - j > k/3$ and $\min\{k, n+1-k\} \leqslant k$. Hence

$$S_3 \leqslant c \sum_{k \in K_3} |k+j|^{-p} \leqslant \begin{cases} c \ln n, & p = 1, \\ c, & p > 1. \end{cases}$$

Finally

$$S_4 \leqslant c \sum_{k \in K_4} \left[\frac{n+1-k}{n^2} \right]^p \leqslant c n^{1-p} \leqslant c.$$

Thus (2.9) follows. ▮

LEMMA 2.2. *Let* (2.2) *and* (2.3) *prevail. If*

$$|\Omega_n'(x_k)| \geqslant c_6 \Delta_n(x_k)^{-1}, \qquad k = 1, 2, \ldots, n, \tag{2.15}$$

then the inequality (1.13) *with* $c = c(c_1, c_2, c_6)$ *holds for every subsequence* $\mathbb{N}^*$.

Proof. Let $x = \cos\theta$, $I = [0, \pi]$,

$$t_{kn} = I \cap [\theta_{kn} - c_1/(4n), \theta_{kn} + c_1/(4n)], \qquad k = 1, 2, \ldots, n;$$

$$I_n = \bigcup_{k=1}^{n} t_{kn};$$

$$h_{kn} = [\theta_{kn}, \theta_{k+1,n}] \setminus I_n, \qquad k = 0, 1, \ldots, n.$$

Claim 1. We have

$$|x-x_{in}| \geqslant c_7 \Delta_n(x_{kn}), \qquad i=k, k+1, \quad \theta \in h_{kn}, \quad k=1, 2, \ldots, n-1,$$
$$|x-x_{1n}| \geqslant c_7 \Delta_n(x_{1n}), \qquad \theta \in h_{0n}, \quad |x-x_{nn}| \geqslant c_7 \Delta_n(x_{nn}), \quad \theta \in h_{nn}.$$

In fact, for $\theta \in h_{kn}$, $1 \leqslant k \leqslant n$, by (2.5) $|x-x_k| \geqslant |\cos(\theta_k + c_1/(4n)) - \cos\theta_k| \geqslant c_7 \Delta_n(x_k)$. Similarly, for $\theta \in h_{kn}$, $0 \leqslant k \leqslant n-1$, we get $|x-x_{k+1}| \geqslant c_7 \Delta_n(x_k)$.

Claim 2. We have

$$|\Omega_n(\cos\theta)| \geqslant c_8, \qquad \theta \in I \setminus I_n. \tag{2.16}$$

We need a result given by Erdős and Turán [2, Lemma IV]

$$\ell_k(x) + \ell_{k+1}(x) \geqslant 1, \qquad x \in [x_{k+1}, x_k], \quad 0 \leqslant k \leqslant n, \quad \ell_0 = \ell_{n+1} = 0. \tag{2.17}$$

Thus

$$\frac{\Omega_n(x)}{\Omega_n'(x_k)(x-x_k)} + \frac{\Omega_n(x)}{\Omega_n'(x_{k+1})(x-x_{k+1})} \geqslant 1,$$
$$x \in [x_{k+1}, x_k], \quad 1 \leqslant k \leqslant n-1. \tag{2.18}$$

Then for $\theta \in h_{kn}$, $1 \leqslant k \leqslant n-1$, by Claim 1 and (2.15)

$$1 \leqslant \frac{c_9}{c_6 \Delta_n(x_k)^{-1}\, c_7 \Delta_n(x_k)} |\Omega_n(x)| = \frac{c_9}{c_6 c_7} |\Omega_n(x)|$$

and hence $|\Omega_n(x)| \geqslant c_8 = c_6 c_7 / c_9$. Similarly, for $\theta \in h_{1n}$ or $\theta \in h_{nn}$, it is enough to use (2.17) with $k=0$ or $k=n$, respectively. This proves (2.16).

Claim 3. Let $I_n' = \bigcup_{k=1}^n t_{kn}'$, where each set t_{kn}' is an interval and $t_{kn}' \cap t_{jn}' = \varnothing$, $k \neq j$, $k, j = 1, 2, \ldots, n$. Then

$$|I_n' \cap I_N| \leqslant \tfrac{7}{8} |I_n'|, \tag{2.19}$$

if N is large enough.

In fact, if $|I_n'| = 0$ then (2.19) is trivial. Now assume that $|I_n'| > 0$ and choose $N > 16 c_1 n / |I_n'|$. We distinguish two cases.

Case 1. $|t_{kn}'| < 2c_1/N$. In this case we use an obvious estimation $|t_{kn}' \cap I_N| \leqslant |t_{kn}'|$.

Case 2. $|t'_{kn}| \geqslant 2c_1/N$. In this case by (2.2) the interval t'_{kn} contains at most

$$\left[|t'_{kn}| \left(\frac{c_1}{N} \right)^{-1} \right] + 1 = \left[\frac{N\,|t'_{kn}|}{c_1} \right] + 1$$

points θ_{iN}'s. Since $|t'_{in}| \geqslant 2c_1/N$, we have

$$\begin{aligned} |t'_{kn} \cap I_N| &\leqslant \left\{ \left[\frac{N\,|t'_{kn}|}{c_1} \right] + 1 \right\} \frac{c_1}{2N} \leqslant \left\{ \frac{N\,|t'_{kn}|}{c_1} + 1 \right\} \frac{c_1}{2N} \\ &\leqslant \frac{|t'_{kn}|}{2} + \frac{c_1}{2N} \leqslant \frac{3}{4}\,|t'_{kn}|. \end{aligned}$$

Thus, recalling $N > 16c_1 n/|I'_n|$,

$$\begin{aligned} |I'_n \cap I_N| &= \left| \bigcup_{k=1}^{n} (t'_{kn} \cap I_N) \right| = \sum_{k=1}^{n} |t'_{kn} \cap I_N| \\ &\leqslant \sum_{|t'_{kn}| < 2c_1/N} |t'_{kn}| + \frac{3}{4} \sum_{|t'_{kn}| \geqslant 2c_1/N} |t'_{kn}| \\ &\leqslant \frac{2c_1 n}{N} + \frac{3}{4}\,|I'_n| \leqslant \frac{1}{8}\,|I'_n| + \frac{3}{4}\,|I'_n| = \frac{7}{8}\,|I'_n|. \end{aligned}$$

This proves Claim 3.

Claim 4. For every fixed integer i the set $I^*_i = \bigcap_{k=i}^{\infty} I_{n_k}$ satisfies

$$|I^*_i| = 0. \tag{2.20}$$

In fact, choosing $I'_n = I_{N_0}$ with $N_0 := n_i$ and applying Claim 3 we conclude that there is a number $N = N_1 \in \mathbb{N}^*$, $N_1 > N_0$, so that $|I_{N_0} \cap I_{N_1}| \leqslant \frac{7}{8}\,|I_{N_0}|$. If $|\bigcap_{k=0}^{j} I_{N_k}| \leqslant (\frac{7}{8})^j\,|I_{N_0}|$ is true for some $j \geqslant 1$ then applying Claim 3 to the set $\bigcap_{k=0}^{j} I_{N_k}$ we can choose a number $N = N_{j+1} \in \mathbb{N}^*$, $N_{j+1} > N_j$, so large that

$$\left| \bigcap_{k=0}^{j+1} I_{N_k} \right| \leqslant \tfrac{7}{8} \left| \bigcap_{k=0}^{j} I_{N_k} \right| \leqslant (\tfrac{7}{8})^{j+1}\,|I_{N_0}|.$$

By induction there is a sequence of integers $n_i = N_0 < N_1 < N_2 < \cdots$, N_1, N_2, $\ldots \in \mathbb{N}^*$, such that

$$\left| \bigcap_{k=0}^{j} I_{N_k} \right| \leqslant (\tfrac{7}{8})^{j}\,|I_{N_0}|$$

holds for every j. Thus

$$|I_i^*| = \left|\bigcap_{k=i}^{\infty} I_{n_k}\right| \leqslant \left|\bigcap_{k=0}^{\infty} I_{N_k}\right| \leqslant \lim_{j\to\infty} \left|\bigcap_{k=0}^{j} I_{N_k}\right| \leqslant \lim_{j\to\infty} \left(\tfrac{7}{8}\right)^j |I_{N_0}| = 0.$$

This proves Claim 4.

For the proof of (1.13) we consider the set

$$I^* = \{\theta \in [0, \pi] : \limsup_{k\to\infty} |\Omega_{n_k}(\cos\theta)| < c_8\}, \tag{2.21}$$

where c_8 is defined in (2.16). Clearly, it suffices to show

$$|I^*| = 0. \tag{2.22}$$

Claim 5. We have

$$I^* \subset \bigcup_{i\in\mathbb{N}^*} I_i^*. \tag{2.23}$$

In fact, $\theta \in I^*$ means $\limsup_{k\to\infty} |\Omega_{n_k}(\cos\theta)| < c_8$, which implies that there is an integer $i = i(\theta) \in \mathbb{N}^*$ such that $|\Omega_{n_k}(\cos\theta)| < c_8$ holds for every $k \geqslant i$. By (2.16), $\theta \in I_{n_k}$ holds for every $k \geqslant i$. That is, $\theta \in I_i^*$. This proves (2.23).

Now (2.22) follows directly from (2.20) and (2.23). ∎

LEMMA 2.3. *We have*

$$\text{(a)} \Leftrightarrow \text{(b)} \Leftrightarrow \text{(c)} \Leftrightarrow \text{(d)} \Rightarrow \text{(e)}.$$

Proof. As we know,

$$\|\Omega_n\| \geqslant \|2T_n\| \geqslant 2, \tag{2.24}$$

where T_n stands for the nth Chebyshev polynomial of the first kind, and by [2, Lemma I]

$$\sum_{k=1}^{n} \frac{1}{|\Omega_n'(x_k)|} \geqslant \frac{1}{4}. \tag{2.25}$$

(a) $\Rightarrow$ (b). By Bernstein Inequality $|\Omega_n'(x_k)| \leqslant c\Delta_n(x_k)^{-1} \|\Omega_n\|$, or equivalently,

$$\|(x - x_k)\, \ell_k(x)\| = \frac{\|\Omega_n\|}{|\Omega_n'(x_k)|} \geqslant c^{-1}\Delta_n(x_k).$$

(b) ⇒ (c). By Statement (b),

$$\|\Omega_n\| \sum_{k=1}^{n} \frac{1}{|\Omega_n'(x_k)|} = \left\| \sum_{k=1}^{n} \left| \frac{\Omega_n(x)}{\Omega_n'(x_k)} \right| \right\| = \left\| \sum_{k=1}^{n} |(x-x_k)\, \ell_k(x)| \right\|$$

$$\leqslant c \sum_{k=1}^{n} \Delta_n(x_k) \leqslant c, \tag{2.26}$$

which coupled with (2.25) yields (1.10). By (1.9), (1.10), and (2.24) the relation (1.11) follows.

(c) ⇒ (d). Inserting $\xi = (x_k + x_{k+1})/2$ into (2.18), we obtain

$$\frac{|\Omega_n(\xi)|}{\frac{1}{2}(x_k - x_{k+1})} \left[\frac{1}{|\Omega_n'(x_k)|} + \frac{1}{|\Omega_n'(x_{k+1})|} \right] \geqslant 1$$

and hence by (1.11) and (2.5)

$$\max_{x \in [x_{k+1},\, x_k]} |\Omega_n(x)| \geqslant |\Omega_n(\xi)| \geqslant \tfrac{1}{2}(x_k - x_{k+1}) \cdot c\Delta_n(x_k)^{-1} \geqslant c.$$

(d) ⇒ (a). By the same argument as that of Lemma 2 in [3].

(a) ⇒ (e). Apply Lemmas 2.1 and 2.2 and use Implication (a) ⇒ (c). ∎

LEMMA 2.4. *If* (1.14) *is true, we have* (2.8), (2.9),

$$\delta_n(x_k) \sim \Delta_n(x_k), \qquad k = 1, 2, \ldots, n, \tag{2.27}$$

and

$$\theta_{k+1,n} - \theta_{kn} \sim \frac{1}{n}, \qquad k = 0, 1, \ldots, n. \tag{2.28}$$

Proof. Clearly

$$\delta_n(x_k) \leqslant \Delta_n(x_k), \qquad k = 1, 2, \ldots, n. \tag{2.29}$$

Hence (1.14) implies (1.8). Then according to Lemma 2.1 the inequalities (2.2), (2.3), (2.8), and (2.9) hold. Applying Lemma 2.3 the relation (1.9) is true. By (1.14) and (1.9)

$$\delta_n(x_k) \geqslant c \frac{\|\Omega_n\|}{|\Omega_n'(x_k)|} \geqslant c\Delta_n(x_k),$$

which coupled with (2.29) gives (2.27). To prove (2.28) it is enough to establish the inequalities $\theta_{1n}-\theta_{0n}\geqslant c/n$ and $\theta_{n+1,n}-\theta_{nn}\geqslant c/n$. By (2.27) with $k=1$ we have $\delta_n(x_1)\geqslant c\Delta_n(x_1)\geqslant c/n^2$ and hence $(1-x_1^2)^{1/2}\geqslant c/n$, which implies $\theta_{1n}-\theta_{0n}=\theta_{1n}\geqslant \sin\theta_{1n}\geqslant c/n$. Similarly we can prove $\theta_{n+1,n}-\theta_{nn}\geqslant c/n$. ∎

LEMMA 2.5. *Let w be a weight on $[-1,1]$ and X the zeros of the orthonormal polynomial $P_n(w,x)$. If (1.16) is true then*

$$w\geqslant cv, \qquad a.e. \tag{2.30}$$

and

$$\gamma_n(w)\sim 2^n. \tag{2.31}$$

Proof. By (1.16), $\lambda_n(x)^{-1}=\sum_{k=0}^{n-1}P_k(x)^2\leqslant cn$ and hence $\lambda_n(x)\geqslant c/n$, which by [5, Theorem 6.2.33, p. 93] gives (2.30). This shows that w belongs to the so-called Szegő class, which implies (2.31) (cf. [5, p. 39]). ∎

LEMMA 2.6. *Let w be a weight on $[-1,1]$ and X the zeros of the orthonormal polynomial $P_n(w,x)$. Then*

$$\text{(A)}\Leftrightarrow\text{(B)}\Leftrightarrow\text{(C)}\Leftrightarrow\text{(D)}\Rightarrow\text{(E)}\Rightarrow\text{(F)}\Leftrightarrow\text{(G)}.$$

Proof. (A) ⇒ (B) By Lemma 2.4 we have (2.27) and (1.8), which gives Statement (B) by Implication (a) ⇒ (b).

(B) ⇒ (C). We need a result given by the author in [6, (14) and (15)]: for an arbitrary weight w

$$|P_n(w,x)|\leqslant c\sum_{k=1}^{n}|(x-x_k)\,\ell_k(x)|, \tag{2.32}$$

which coupled with (1.15) yields (1.16). Besides the relation (1.17) follows from (1.15), (1.16), (2.2 4), and (2.31).

(C) ⇒ (D). The inequality (1.18) may be deduced from (2.17) by the same argument as that of Implication (c) ⇒ (d).

(D) ⇒ (A). Apply Lemma 2 in [3] and use (2.31).

(A) ⇒ (E). The inequality (1.19) follows from (1.13) and (2.31).

(E) ⇒ (F). By Lemma 2.5 we get (2.30). On the other hand, by [4, (10.3)]

$$\lim_{n\to\infty}\int_{-1}^{1}\left|[P_n(x)^2-2xP_n(x)\,P_{n-1}(x)+P_{n-1}(x)^2]\,w(x)-\frac{2}{\pi}(1-x^2)^{1/2}\right|dx=0.$$

According to the result that strong convergence in L_1 implies a.e. convergence for a subsequence we get

$$[P_{n_k}(x)^2 - 2xP_{n_k}(x)\, P_{n_k-1}(x) + P_{n_k-1}(x)^2]\, w(x) \to \frac{2}{\pi}(1-x^2)^{1/2}, \qquad \text{a.e.}, \quad k \to \infty.$$

We rewrite the above relation as

$$\{[P_{n_k-1}(x) - xP_{n_k}(x)]^2 + (1-x^2)\, P_{n_k}(x)^2\}\, w(x) \to \frac{2}{\pi}(1-x^2)^{1/2}, \qquad \text{a.e.}, \quad k \to \infty.$$

Hence

$$\limsup_{k\to\infty}\, (1-x^2)\, P_{n_k}(x)^2\, w(x) \leqslant \frac{2}{\pi}(1-x^2)^{1/2}, \qquad \text{a.e.}$$

Using (1.19) we get $w \leqslant cv$.

(F) ⇒ (G). Use $\lambda_n(v, x) \sim 1/n$ and an equivalent definition of $\lambda_n(w, x)$:

$$\lambda_n(w, x) = \min_{Q \in \mathbf{P}_{n-1}} \int_{-1}^{1} \left[\frac{Q(t)}{Q(x)}\right]^2 w(t)\, dt \sim \min_{Q \in \mathbf{P}_{n-1}} \int_{-1}^{1} \left[\frac{Q(t)}{Q(x)}\right]^2 v(t)\, dt = \lambda_n(v, x) \sim \frac{1}{n}.$$

(G) ⇒ (F). By [5, Theorem 6.2.33, p. 93] we have

$$\pi(1-x^2)^{1/2}\, w(x) \geqslant \limsup_{n\to\infty}\, n\lambda_n(w, x) \geqslant c$$

and hence we can apply [5, Theorem 6.2.34, p. 93] to get

$$\pi(1-x^2)^{1/2}\, w(x) = \lim_{n\to\infty} n\lambda_n(w, x) \sim 1. \quad \blacksquare$$

LEMMA 2.7. *Let $m \in \mathbb{N}_2$. If (1.8) is true and $m-j$ is odd then*

$$\left\| \sum_{k=1}^{n} |A_{jk}(X)| \right\| \leqslant \frac{c \ln n}{n^j}. \tag{2.33}$$

Proof. By Lemma 2.1 we obtain (2.2), (2.3), and (2.8), the last relation of which gives (2.6). Then we can apply Lemma C to get (2.7). Thus by (1.8), (2.7), and (2.8)

$$\begin{aligned}|A_{jk}(x)| &\leqslant c \sum_{i=0}^{m-j-1} |b_{ik}(x-x_k)^{i+j}\,\ell_k(x)^m| \\ &= c \sum_{i=0}^{m-j-1} |b_{ik}|\,|(x-x_k)\,\ell_k(x)|^{i+j}\,|\ell_k(x)|^{m-i-j} \\ &\leqslant c \sum_{i=0}^{m-j-1} \Delta_n(x_k)^{-i}\,\Delta_n(x_k)^{i+j}\,|\ell_k(x)|^{m-i-j} \\ &\leqslant c\Delta_n(x_k)^j \sum_{i=0}^{m-j-1} |\ell_k(x)|^{m-i-j} \\ &\leqslant c\Delta_n(x_k)^j\,|\ell_k(x)|.\end{aligned}$$

Thus according to (2.9)

$$\left\| \sum_{k=1}^{n} |A_{jk}| \right\| \leqslant c \left\| \sum_{k=1}^{n} \Delta_n(x_k)^j\,|\ell_k| \right\| \leqslant \frac{c}{n^j} \left\| \sum_{k=1}^{n} |\ell_k| \right\| \leqslant \frac{c \ln n}{n^j}. \quad \blacksquare$$

LEMMA 2.8. *If a pointsystem X satisfies* (1.6) *and* (1.7) *then* (1.8) *holds.*

Proof. Using the relation (2.38) of [7]

$$\sum_{k=1}^{n} |(x-x_k)\,A_{1k}(x)| \leqslant \sum_{k=1}^{n} (x-x_k)^2\,A_{0k}(x), \qquad x \in \mathbb{R},$$

by (1.6) we obtain

$$\sum_{k=1}^{n} |(x-x_k)\,A_{1k}(x)| \leqslant 4 \sum_{k=1}^{n} |A_{0k}(x)| \leqslant 4nc_m.$$

By means of (2.1) with $i=m-1$ and $j=1$ for each k, $1 \leqslant k \leqslant n$,

$$4nc_m \geqslant |(x-x_k)\,A_{1k}(x)| \geqslant (x-x_k)^2 \left(\frac{x-x_k}{d_k}\right)^{m-2} \ell_k(x)^m$$

and hence

$$|(x-x_k)\,\ell_k(x)| \leqslant (4nc_m)^{1/m}\,d_k^{(m-2)/m}.$$

Thus

$$|(x-x_k)\,\ell_k(x)| \leqslant \limsup_{m\to\infty} [(4nc_m)^{1/m}\,d_k^{(m-2)/m}] \leqslant cd_k. \tag{2.34}$$

On the other hand, (1.6) with $m=2$ [1, Sect. 4] implies (2.2) and (2.3), which yields (2.5). Hence (2.34) means (1.8). ▮

LEMMA 2.9. *If*

$$\left\| \sum_{k=1}^{n} |A_{0knm}| \right\| \leqslant c_m \tag{2.35}$$

holds for every $m \in \mathbb{N}_2$ *and* (1.7) *is true then* (1.21) *holds for every* $m \in \mathbb{N}_2$.

Proof. Clearly, (2.35) implies (1.6). Then applying Lemma 2.8 we get (1.8). According to Lemma 2.7 we have the estimation (2.33), which coupled with (2.35) by [7, Theorem 4.3] proves our conclusion. ▮

3. PROOFS OF THEOREMS

3.1. *Proof of Theorem* 1.1. Apply Lemmas 2.8 and 2.3. ▮

3.2. *Proof of Theorem* 1.2. According to Theorem 1.1 the inequalities (1.8) and (1.13) are true. By means of (2.32) we obtain (1.16) and hence by Lemma 2.5 get (2.31). Thus (1.19) follows from (1.13) and (2.31). Therefore Statement (E) is true and by Lemma 2.6 Statement (F) is valid. Further, in virtue of [5, Lemma 6.3.6, p. 108] (2.28) is true and hence (2.27) occurs. This shows Statement (A). It remains to apply Lemma 2.6. ▮

3.3. *Proof of Theorem* 1.3. Since $\sum_{k=1}^{n} A_{0k}(x) \equiv 1$, the relation (1.5) implies $c_m \equiv 1$ in (2.35). Then apply Theorems 1.1 and 1.2 and Lemma 2.9. ▮

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